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A numerical study of free convection in a strongly inhomogeneous gas medium†

I. M. KOZLOV, A. Ye. SUVOROV and V. I. TYUKAYEV

Academic Scientific Complex, A. V. Luikov Heat and Mass Transfer Institute, Minsk 220072, Belarus

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Abstract—The equations for free convection in gas are analyzed and the computational algorithm, efficient when the condition $\mu gh/RT \ll 1$ is met at an arbitrary (not small) temperature drop, is proposed. Free convection arising in a steady nonuniform heating of walls is calculated for axisymmetric objects of various shapes. The calculated results demonstrate the establishment of steady and periodic gas flows.

1. INTRODUCTION

A standard approximation in studying convection is the Boussinesq approximation [1–4] with which density variations are assumed to be so small that the condition $\delta\rho/\rho \ll 1$ is met, i.e. the medium is, in effect, taken to be incompressible. This is a good approximation for liquids, whereas, for gases, it imposes a substantial limitation on the temperature drop that should also satisfy the relation $\delta T/T \ll 1$. At an arbitrary temperature drop, density variations are not small and in this sense the gas medium cannot be regarded as incompressible. Yet, the solution of the complete system of gasdynamic equations, including that describing also the propagation of sound waves, is impossible because typical velocities in convection are smaller than the sound velocity by orders of magnitude, and the temporal step of numerical schemes is restricted by the most rapid process that this scheme accounts for.

Simple evaluations reveal that, at low velocities of motion, pressure in the system is nearly invariable spatially and can be presented as $p(\mathbf{x}, t) = p_0(t) + \hat{p}(\mathbf{x}, t)$, where $p_0(t)$ is the spatially-constant quantity dependent on time owing to the total heating or cooling of the system and $\hat{p}(\mathbf{x}, t)$ is small. This small correction to pressure can be taken into account in the motion equation only, whereas a local gas pressure is defined by the quantity $p_0(t)$ and by the local temperature with the aid of the equation of state. The condition of compatibility of the motion and continuity equations allows a formulation of the equation for $\hat{p}(\mathbf{x}, t)$. The elliptic character of this equation indicates that there are no sound waves in the gas medium thus described, which precludes the above-mentioned difficulties associated with the choice of the temporal step. Essentially, the proposed method is an extension to gas media of the familiar MAC method [5] widely applied for calculating the motion of incompressible liquids.

Section 2 of the present article reports the analysis of the equations for convective motion of the gas at arbitrary temperature drops, Section 3 outlines the numerical scheme and Section 4 gives the calculated results.

2. EQUATIONS OF CONVECTION IN THE GAS MEDIUM

The gas convection is described by gas dynamic equations with regard to viscosity and thermal conductivity [1]:

$$\frac{D\rho}{Dt} = -\rho \operatorname{div}(\mathbf{v}) \tag{1}$$

$$\rho \frac{D\mathbf{v}}{Dt} = \rho g - \nabla p + \mathbf{Q} \tag{2}$$

$$\rho c_p \frac{DT}{Dt} = \beta T \frac{Dp}{Dt} + \operatorname{div}(\kappa \nabla T) + \Phi \tag{3}$$

where ρ is the gas density, $\mathbf{v}$ is the gas velocity, T is the temperature, g is the acceleration due to gravity, κ is the thermal conductivity, c_p is the specific heat and $\beta = -(\partial\rho/\partial T)_p/\rho$ is the thermal expansion. The gas pressure p is found from ρ and T using the equation of state

$$p = p(\rho, T). \tag{4}$$

The terms related to viscosity are defined by the following expressions [2]

$$Q_i = \frac{\partial}{\partial x_k} \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \tfrac{2}{3}\delta_{ik} \frac{\partial v_l}{\partial x_l} \right) \tag{5}$$

$$\Phi = \eta \left(\frac{\partial v_l}{\partial x_k} + \frac{\partial v_k}{\partial x_l} - \tfrac{2}{3}\delta_{lk} \frac{\partial v_m}{\partial x_m} \right) \frac{\partial v_l}{\partial x_k} \tag{6}$$

where η is the viscosity, and the repetitive subscripts

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NOMENCLATURE

c	sound velocity
c_p	specific heat
g	acceleration due to gravity
p	pressure
R	universal gas constant
$\mathfrak{R}$	Rayleigh number
t	time
T	temperature
v	velocity
x	coordinate.

Greek symbols

β	thermal expansion
η	viscosity
κ	thermal conductivity
μ	atomic weight
ν	kinematic viscosity
ρ	density
χ	thermal diffusivity.

k, l and m imply summation. The system of equations is closed by the boundary conditions

$$v_i|_{\text{at the boundary}} = 0$$

$$T|_{\text{at the boundary}} = T(t). \tag{7}$$

With the aim to simplify the initial equations, simple estimations will be made assuming that convection proceeds in a gas medium. Provided $\Delta\rho$ is the characteristic density variation due to heating, while rising to a height h, the heated gas acquires the velocity of the order of $v \sim \sqrt{(gh\Delta\rho/\rho)}$ owing to the buoyancy. In fact, by virtue of the effects of viscosity and cold gas indraft that retard the motion, the above-mentioned velocity can be appreciably lower, but this justifies our estimations even to a greater extent. With a fulfillment of the condition

$$\varepsilon = \frac{\mu g h}{RT} \ll 1 \tag{8}$$

this velocity appears to be much lower than the sound velocity $c \sim \sqrt{(\mathrm{RT}/\mu)}$. In the above equations, μ is the atomic weight and R is the universal gas constant. Under the same conditions, the hydrostatic pressure $p_h \sim \rho g h$ and the dynamic pressure $p_d \sim \rho v^2$ prove to be much lower than the total pressure

$$p_d \leqslant p_h \ll p. \tag{9}$$

Thus, to a high degree of accuracy, pressure is equal to

$$p(\mathbf{x}, t) = p_0(t) + \hat{p}(\mathbf{x}, t). \tag{10}$$

Here $\hat{p} \ll p_0$ and a time dependence of p_0 is determined by the total heating of the gas. Since condition (8) in laboratory conditions is fulfilled with good assurance, the initial equations can be simplified by retaining the correction to pressure in motion equation (2) alone. With the same accuracy, the energy dissipation due to viscosity in heat conduction equation (3) can be disregarded. The indicated manipulations result in

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\,\mathrm{div}\,(\mathbf{v}), \tag{11}$$

$$\rho\frac{\mathrm{D}v}{\mathrm{D}t} = (\rho - \rho_0)g - \nabla\hat{p} + \mathbf{Q} \tag{12}$$

$$Q_i = \frac{\partial}{\partial x_k}\eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) \tag{13}$$

$$\rho c_p\frac{\mathrm{D}T}{\mathrm{D}t} = \beta T\frac{\mathrm{d}p_0}{\mathrm{d}t} + \mathrm{div}\,(\kappa\nabla T) \tag{14}$$

$$p_0(t) = p(\rho, T) \tag{15}$$

where ρ_0 is the volume-average gas density†.

The rearrangements performed have far reaching effects. Primarily, it is seen from equation (4) that the gas density is unambiguously determined by temperature and by the equation of state of the gas, p_0 being defined here by the law of mass conservation in the volume

$$M = \int \rho\,\mathrm{d}V. \tag{16}$$

The correction to the pressure $\hat{p}$ does not enter into the equation of state and should be ascertained from other considerations. Namely, by calculating the divergence from equation (12) we can obtain an elliptic equation for $\hat{p}$. The method for solving these equations will be examined in more detail later, now we will point out only the most important consequence of the performed approximations, namely sound waves do not propagate in the medium described by equations (11)–(15). Since $v \ll c$, this has no physical meaning whatsoever but plays a prominent role from the viewpoint of applying numerical methods.

With small temperature drops, $T' = T - T_0 \ll T$, further simplifications of the system of equations are possible. In this case, $\rho - \rho_0 = -\rho_0\beta T' \ll \rho_0$. Dis-

† In equation (12) we carried out the substitution

$$\hat{p} \to \hat{p} + \rho_0 g\cdot\mathbf{x} - \eta\frac{2}{3}\frac{\partial v_l}{\partial x_l}$$

where ρ_0 is the volume-average gas density that is, evidently, independent of time. Such substitution somewhat simplifies the equation form. Furthermore, the very possibility of arbitrary redefinition, via such substitution, of a diagonal part of the viscous stress tensor implies that, with the approximations made, the second viscosity coefficient [2], of which no account was taken in the equation, entirely drops out of the solution.

carding the density variation in heating everywhere except the buoyancy term, setting κ and η to be constant over the volume, and omitting the index at ρ_0, we obtain the system of equations

$$\nabla \cdot \mathbf{v} = 0, \tag{17}$$

$$\frac{D\mathbf{v}}{Dt} = -\nabla \frac{\hat{p}}{\rho} + \nu \Delta \mathbf{v} - \beta g T' \tag{18}$$

$$\frac{DT'}{Dt} = \chi \Delta T' \tag{19}$$

where $\nu = \eta/\rho$ is the kinematic viscosity and $\chi = \kappa/\rho c_p$ is the thermal diffusivity. The convection equations of the form (17)–(19) are known as the Boussinesq approximation [1–4] and are used extensively for analyzing convection in liquids. A qualitative analysis of equations (17)–(19) shows [2, 6] that the flow character is defined in this case by the Rayleigh number

$$\Re = \frac{g\beta T' h^3}{\nu \chi} \tag{20}$$

where h is the characteristic volume dimension.

3. NUMERICAL METHOD OF SOLVING EQUATIONS

The applied numerical methods are essentially different for incompressible liquids and for gases. Although there are plenty of various methods for solving gasdynamic equations (1)–(4), they are identical in essence. Should all flow characteristics at a certain time instant t be known, then, after the forces acting on each element of the medium being computed with the aid of equation of state (4), the flow characteristics at the time instant $t+\Delta t$ can be obtained. All these methods involve a restriction on the temporal step referred to as the Courant criterion [5]

$$\Delta t < \frac{\Delta x}{c+v} \tag{21}$$

where Δx is the dimension of spatial cells and c is the sound velocity. As applied to the problems of convection in gas media, this criterion is very stringent because characteristic velocities, observable in convection, are lower than the sound velocity by orders of magnitude.

For predicting the liquid motion, the MAC method [5, 9] is known, using which pressure can be eliminated from the system of equations (17)–(19). The essence of this method is the following: if it is supposed that pressure at all flow points at the time instant t is known, then, in much the same manner as it is done for gas media, we can compute the liquid velocities at the time instant $t+\Delta t$ with the aid of equation (18). Be it now required that these velocities comply with continuity equation (17), an elliptic closed equation for pressure will be derived. Having solved this equation we can actually find velocities at the time instant $t+\Delta t$. As it turns out [5, 9], the sound velocity drops out of the criterion for the temporal step for this method, equation (21), which is a consequence of the above-stated absence of sound waves from the solution to equations (11)–(15) and (17)–(19). The present study generalizes this method for calculating slow ($v \ll c$) axisymmetric flows of a compressible gas with allowance for thermal conductivity, viscosity, and gravity force that are described by equations (11)–(15). A similar method for plane flows is delineated in refs. [7, 8].

If we know all gas parameters at the time instant t^n, the parameters at the time instant $t^{n+1} = t^n + \Delta t$ are calculated in two stages. First, T^{n+1} is computed throughout the volume using the solution for the heat conduction equation

$$\rho^n c_p^n \left(\frac{T^{n+1} - T^n}{\Delta t} + \mathbf{v}^n \cdot \nabla T^{n+1} \right) = \beta^n T^n \left(\frac{dp_0}{dt} \right)^n + \operatorname{div}(\kappa^n \nabla T^{n+1}). \tag{22}$$

Should we restrict ourselves to the equation of state of the ideal gas

$$p = \frac{\rho R T}{\mu} \tag{23}$$

then

$$p_0(t) = \frac{MR}{\mu} \left(\int \frac{dV}{dT} \right)^{-1}. \tag{24}$$

Hence

$$\frac{dp_0}{dt} = \frac{\gamma - 1}{M} \int \rho \operatorname{div}(\kappa \nabla T)\, dV \tag{25}$$

where $\gamma = c_p/c_v$ is the adiabatic exponent. Having thus determined temperature and density at a new temporal stratum, we compute velocity. Let us write continuity equation (10) by the implicit scheme

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} = -\operatorname{div}(\rho^{n+1}\mathbf{v}^{n+1}) \tag{26}$$

and motion equation (12), by the explicit scheme

$$\frac{\rho^{n+1}\mathbf{v}^{n+1} - \rho^n \mathbf{v}^n}{\Delta t} = (\rho^n - \rho_0) g - \nabla \hat{p} + \mathbf{Q}^n - \mathbf{S}^n \tag{27}$$

where $\mathbf{S} = (\nabla \cdot \mathbf{v}^n)\rho^n \mathbf{v}^n$. The condition of compatibility of the two equations yields the equation for the unknown $\hat{p}$

$$\Delta \hat{p} = \frac{\rho^{n+1} - \rho^n}{\Delta t^2} + \operatorname{div}\left(\frac{\rho^n \mathbf{v}^n}{\Delta t} + \rho^n g + \mathbf{Q}^n - \mathbf{S}^n \right). \tag{28}$$

If we resort once more to equation (27), it is possible to impart a more symmetrical form to the pressure equation

$$\Delta \hat{p} = \frac{\rho^{n+1} - 2\rho^n + \rho^{n-1}}{\Delta t^2} + \operatorname{div}(\rho^n g + \mathbf{Q}^n - \mathbf{S}^n). \tag{29}$$

Having solved equation (28) we can find the velocities

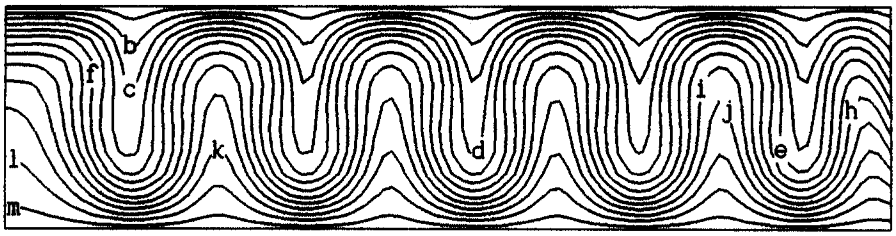

Fig. 1. Stationary temperature distribution: (a) 301 K, (b) 323 K, (c) 346 K, (d) 368 K, (e) 390 K, (f) 413 K, (g) 435 K, (h) 457 K, (i) 480 K, (j) 502 K, (k) 524 K, (l) 546 K and (m) 569 K.

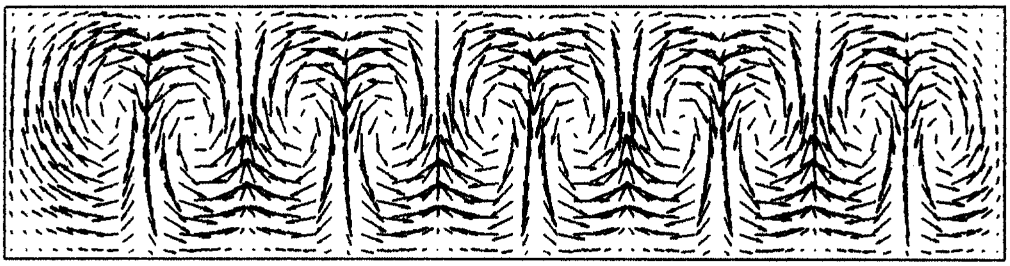

Fig. 2. Stationary velocity field (a maximal velocity is 0.1 m s^{-1}).

on a new temporal stratum using equation (27), which completes calculations at one temporal step.

Equations (22), (27) and (28) are quantized with respect to spatial variables with the help of a spaced grid with rectangular meshes [9]. The resulting five-point equations for pressure and temperature are solved using the "α–β" iteration algorithm [10].

4. CALCULATED RESULTS

By the outlined algorithm, calculations of convection in argon were carried out that attested to a high efficiency of the algorithm. For calculations use was made of the equation of state of the ideal gas, as well as the viscosity and the thermal conductivity coefficients that are proportional to $\sqrt{T}$. Absolute values of the coefficients were taken in conformity with ref. [11].

Consider convection in argon located in a cylindrical plane volume of radius 10 cm and height 1 cm. The initial pressure of argon is 0.06 MPa and temperature, 290 K. The volume bottom is held at a temperature of 580 K, and, at the volume walls, a linear distribution of temperature ranging from 580 to 290 K is maintained. An initially stationary gas begins to move as it gets heated: first, a vortex forms

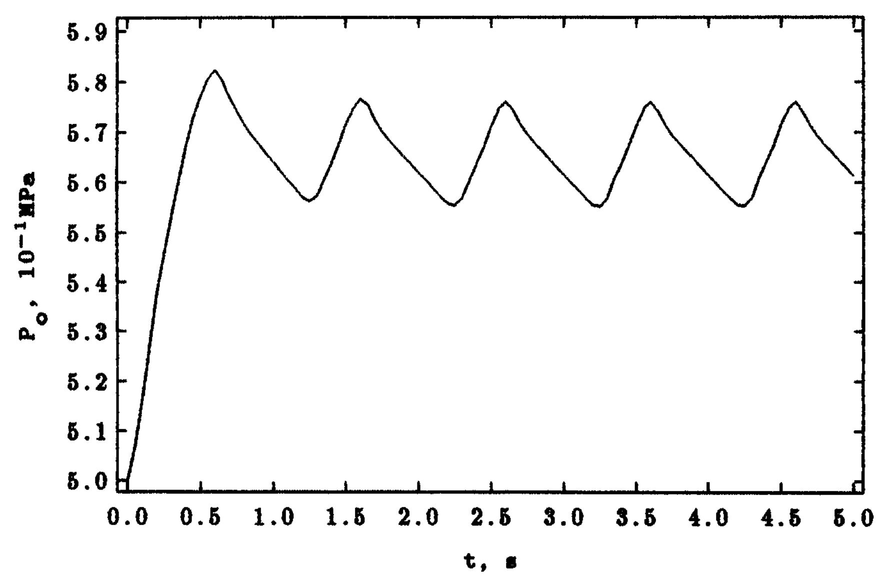

Fig. 3. Time dependence of pressure in the volume.

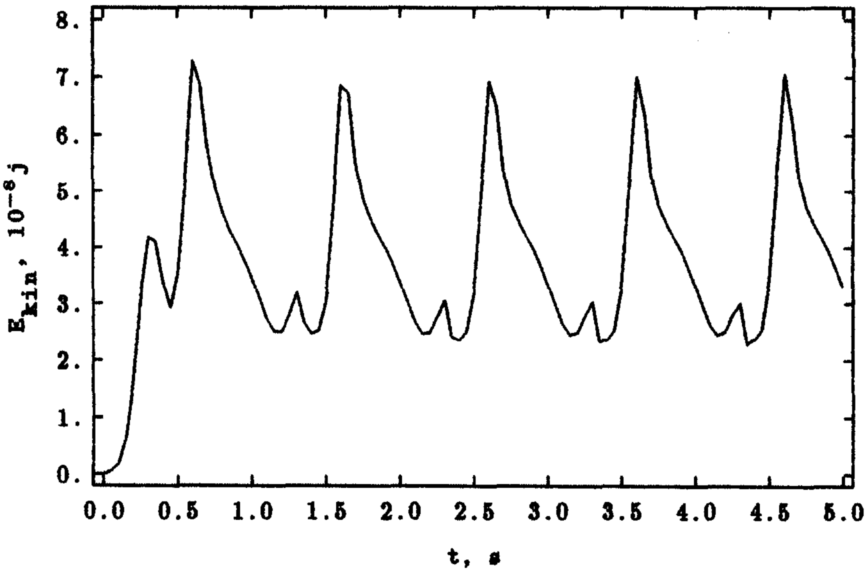

Fig. 4. Time dependence of kinetic energy in the volume.

near the wall along which the heated gas rises. The motion upsets an unsteady equilibrium, and nine more vortexes form one by one. After a lapse of time the motion becomes steady. Figures 1 and 2 demonstrate stationary temperature distributions and velocity fields. A similar flow, observed experimentally in silicone oil, is described in ref. [12].

The second calculation was performed for a volume made up of two cylinders of radii 0.5 cm and 1 cm and heights 2 and 1 cm, respectively. A thin cylinder

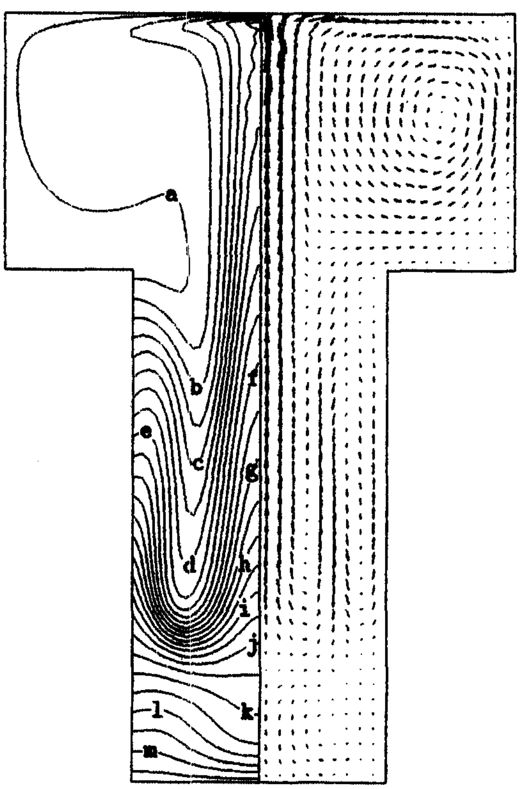

Fig. 5. Temperature and velocity fields at the time instant $t = 3$ s: (a) 301 K, (b) 323 K, (c) 346 K, (d) 368 K, (e) 390 K, (f) 413 K, (g) 435 K, (h) 457 K, (i) 480 K, (j) 502 K, (k) 524 K, (l) 546 K and (m) 569 K.

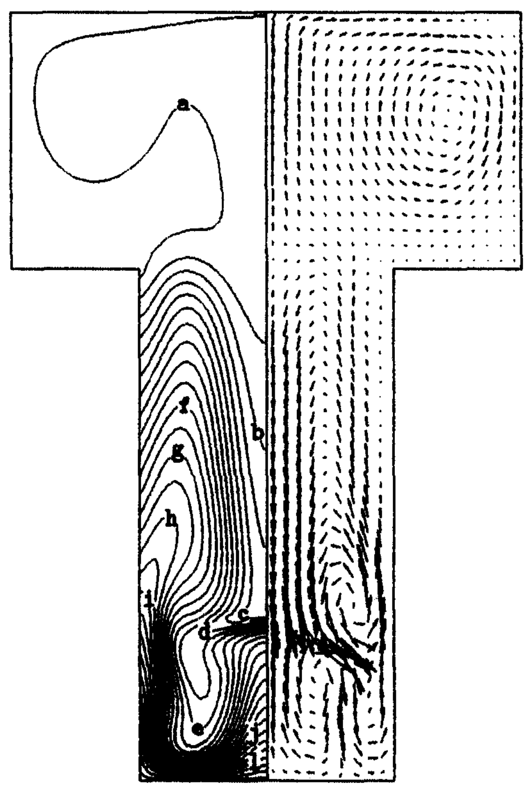

Fig. 6. Temperature and velocity fields at the time instant $t = 3.3$ s: (a) 301 K, (b) 323 K, (c) 346 K, (d) 368 K, (e) 390 K, (f) 413 K, (g) 435 K, (h) 457 K, (i) 480 K, (j) 502 K, (k) 524 K, (l) 546 K and (m) 569 K.

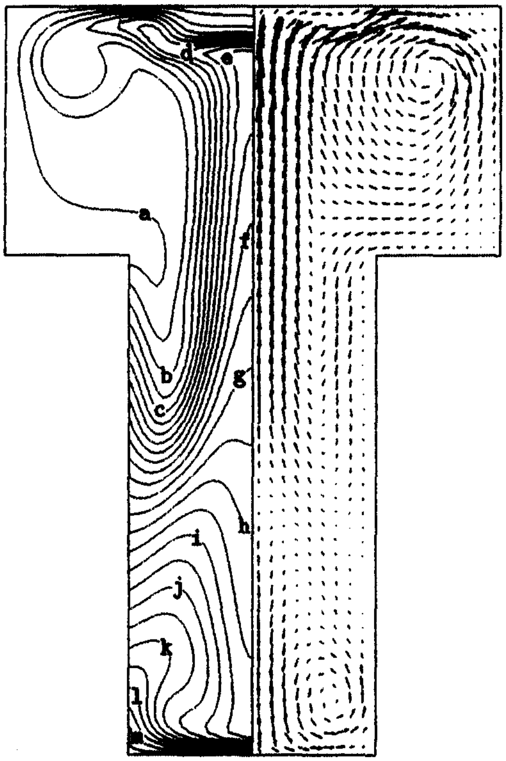

Fig. 7. Temperature and velocity fields at the time instant $t = 3.6$ s: (a) 301 K, (b) 323 K, (c) 346 K, (d) 368 K, (e) 390 K, (f) 413 K, (g) 435 K, (h) 457 K, (i) 480 K, (j) 502 K, (k) 524 K, (l) 546 K and (m) 569 K.

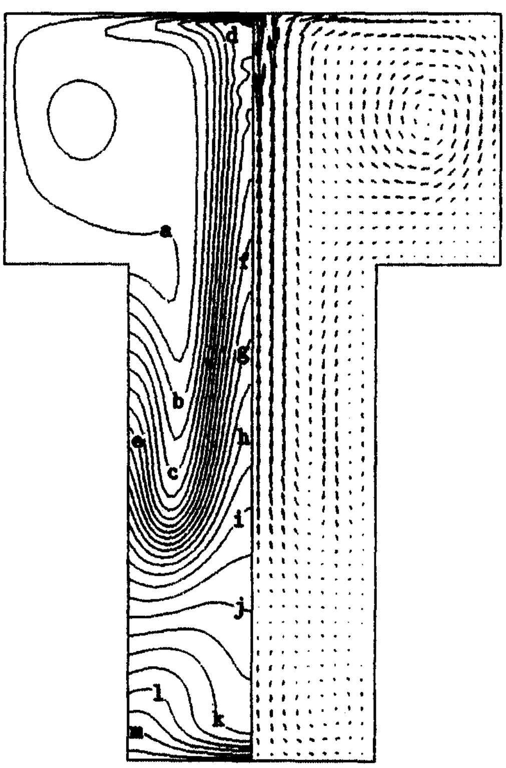

Fig. 8. Temperature and velocity fields at the time instant $t = 3.8$ s: (a) 301 K, (b) 323 K, (c) 346 K, (d) 368 K, (e) 390 K, (f) 413 K, (g) 435 K, (h) 457 K, (i) 480 K, (j) 502 K, (k) 524 K, (l) 546 K and (m) 569 K.

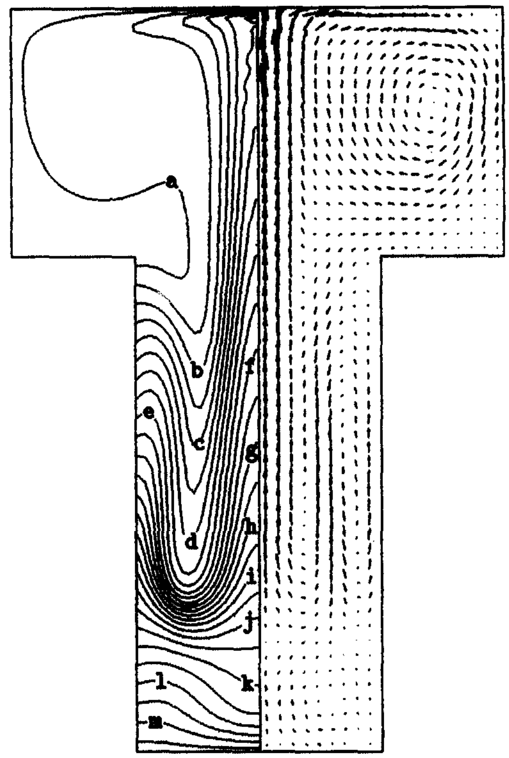

Fig. 9. Temperature and velocity fields at the time instant $t = 4$ s: (a) 301 K, (b) 323 K, (c) 346 K, (d) 368 K, (e) 390 K, (f) 413 K, (g) 435 K, (h) 457 K, (i) 480 K, (j) 502 K, (k) 524 K, (l) 546 K and (m) 569 K.

was positioned below, with its bottom kept at a temperature of 580 K and a linear temperature distribution from 580 to 290 K maintained on its wall, whereas all surfaces of a broader cylinder were held at 290 K. The initial pressure of argon was 0.5 MPa. Figures 3 and 4 show time dependencies for $p_0(t)$ and for the total kinetic energy of the gas. The character of the dependences clearly indicates that, in the conditions considered, a periodic gas motion with a period close to 1 s is established. Figures 5–9 give temperature and velocity fields at various time instants within one period. It is evident from the figures that the flow character is determined by a periodic penetration of the cold gas into the region of the working cell, its heating, and a subsequent ejection into the cold region. Comparing Figs. 5 and 9 reveals that the flows at these time instants are actually identical.

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